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A Geometric Theory of Surface Area

Part III: Non-triangulable Parametric Surfaces

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Introduction

In Part I [1] we presented a geometric theory of the area of a non-parametric surface. In Part II [2] we considered the area of a triangulable parametric surface. To round out the basic theory we now take up the case of the non-triangulable parametric surfaces.

A parametric surface is the locus in $\mathfrak{E}^3$ of simultaneous equations $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$, these functions being defined and continuous on $\mathfrak{E}$, a subset of the uv plane consisting of the interior and the boundary of a simple closed polygon. These equations constitute a continuous transformation or mapping F of $\mathfrak{E}$. Such a surface S is said to be triangulable at a given point $Q \in S$, if for every ball $B(Q, \varepsilon)$, there exists an admissible triangle T inscribed in $S \cap B(Q, \varepsilon)$ (i. e., the vertices of T are in $S \cap B(Q, \varepsilon)$ and one angle of T lies between a prescribed angle φ, $0 < \varphi < \pi$, and $\pi - \varphi$). S is said to be triangulable if it is triangulable at each of its points.

1. Topology

Let S be non-triangulable at $Q \in S$. There exists a ball $B(Q, \varepsilon)$ such that in $S \cap B(Q, \varepsilon)$ no admissible triangle can be inscribed. It follows that S is non-triangulable at each of the points in $S \cap B(Q, \varepsilon)$. Let D denote the set of the points of S at which S is non-triangulable. It is seen that D is open relative to S. Since S is a continuous map of $\mathfrak{E}$, no point of D is an isolated point (unless S itself consists of only one point).

We now consider the components (maximal connected subsets) of D. Let C denote one such component. Since C is open relative to S, it follows that for every $Q \in C$, if $\varepsilon > 0$ is sufficiently small, the boundary of $B(Q, \varepsilon)$ relative to S, is of dimension zero [3]. Thus C is of dimension 1 at every one of its points. Since C consists of more than one point, is connected and contains no trees (i. e., no subset of C is a tree), it follows [4, 5] that $\bar{C}$, the closure of C, is either a Jordan curve or an arc. If $\bar{C}$ is a Jordan curve, it can be decomposed as the union of two arcs. We may then consider $\bar{D}$ as the union of a set of arcs. If A is such an arc, we may set up an order relation [4] on A which is isomorphic to the natural order relation on the closed linear interval [0, 1].

2. Surface Area

Theorem 1.

$\bar{C}$ can be imbedded in a triangulable surface.

Proof:

For each point Q of $\bar{C}$, there exists a ball $B(Q, \varepsilon_Q)$ such that $C \cap B(Q, \varepsilon_Q)$ consists of only one component. Let Q range over $\bar{C}$. This gives us a covering of $\bar{C}$. Since $\bar{C}$ is connected, there exists [4] a simple chain of these balls which connect the two end-points of $\bar{C}$. There exists an arc k every point of which lies on the boundary of this finite set of balls.

We associate the arc k to the arc $\bar{C}$. It is seen that it is possible to select arcs k in such a manner that if $\bar{C}_1$ and $\bar{C}_2$ have a common end-point, then their corresponding arcs k_1 and k_2 also have a common end-point.

In the isomorphism of k and $\bar{C}$, let $M \in k$ and $M' \in \bar{C}$ be two corresponding points. Consider the line segment joining M and M'. Prolong this segment beyond M' so that M' becomes the mid-point of the extended segment. The union of these extended segments (as point sets) constitute a 2-dimensional strip which, clearly, is a triangulable surface.

Thus, it is clear that given any non-triangulable surface, S, there exists a triangulable surface S^* of which S is a subset. We now define the area of S to be the G. L. B. of the set of the areas of all the triangulable surfaces which contain S as a subset.

Each extension S^* of S as described above, involves an extension $\mathfrak{C}^*$ of $\mathfrak{C}$ and a corresponding extension F^* of the mapping F.

It is easy to see that, for a given S^* one may choose $\mathfrak{E}^*$ to be an admissible set in the uv plane, i. e., a set which consists of the interior and the boundary of a simple closed polygon.

Theorem 2.

Let S be a non-triangulable surface. The area of S, as above defined, is identical with its Lebesgue area.

Proof:

Let A denote the area of S. There exists a sequence $(S_1^*, S_2^*, \ldots)$ of triangulable surfaces each containing S as a subset and such that

1) the corresponding sequence $(A_1^*, A_2^*, \ldots)$ of the surface areas converges to A,

2) the corresponding sequence $(\mathfrak{E}_1^*, \mathfrak{E}_2^*, \ldots)$ is monotonic decreasing, i. e., $\mathfrak{E}_1^* \supset \mathfrak{E}_2^*, \ldots$, and

3) $\mathfrak{E} = \bigcap_1^\infty \mathfrak{E}_i^*$.

It was shown in [2] that the area of each S_i^* is identical with its Lebesgue area. It now follows [2] from the convergence theorem for Lebesgue areas that the Lebesgue area of S is precisely A.

References

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